INITIAL ALGEBRAS OF DETERMINANTAL RINGS, COHEN-MACAULAY AND ULRICH IDEALS

WINFRIED BRUNS, TIM RÖMER, AND ATTILA WIEBE

ABSTRACT. We study initial algebras of determinantal rings, defined by minors of generic matrices, with respect to their classical generic point. This approach leads to very short proofs for the structural properties of determinantal rings. Moreover, it allows us to classify their Cohen-Macaulay and Ulrich ideals.

1. INTRODUCTION

Let K be a field and X an $m \times n$ matrix of indeterminates over K. Let $K[X]$ denote the polynomial ring generated by all the indeterminates X_{ij}. For a given positive integer $r \leq \min\{m,n\}$ we consider the determinantal ideal $I_{r+1} = I_{r+1}(X)$ generated by all $r+1$ minors of X if $r < \min\{m,n\}$ and $I_{r+1} = (0)$ otherwise. Let $R_{r+1} = R_{r+1}(X)$ be the determinantal ring $K[X]/I_{r+1}$.

Determinantal ideals and rings are well-known objects and the study of these objects has many connections with algebraic geometry, invariant theory, representation theory and combinatorics. See Bruns and Vetter [BV] for a detailed discussion.

In the first part of this paper we develop an approach to determinantal rings via initial algebras. We cannot prove new structural results on the rings R_{r+1} in this way, but the combinatorial arguments involved are extremely simple. They yield quickly that R_{r+1}, with respect to its classical generic point, has a normal semigroup algebra as its initial algebra. Using general results about toric deformations and the properties of normal semigroup rings, one obtains immediately that R_{r+1} is normal, Cohen-Macaulay, with rational singularities in characteristic 0, and F-rational in characteristic p.

Toric deformations of determinantal rings have been constructed by Sturmfels [St] for the coordinate rings of Grassmannians (via initial algebras) and Gonciulea and Lakshmibai [GL] for the class of rings considered by us. The advantage of our approach, compared to that of [GL], is its simplicity.

Moreover, it allows us to determine the Cohen-Macaulay and Ulrich ideals of R_{r+1}. Suppose that $1 \leq r < \min\{m,n\}$ and let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the ideal of R_{r+1} generated by the r-minors of the first r rows (resp. the first r columns) of the matrix X. The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of height one and hence they are divisorial, because R_{r+1} is a normal domain. The divisor class group $\mathrm{Cl}(R_{r+1})$ is isomorphic to $\mathbb{Z}$ and is generated by the class $[\mathfrak{p}] = -[\mathfrak{q}]$. (See Bruns and Herzog [BH, Section 7.3] or [BV, Section 8].) The symbolic powers of $\mathfrak{p}$ and $\mathfrak{q}$ coincide with the ordinary ones. Therefore the ideals $\mathfrak{p}^k$ and $\mathfrak{q}^k$ represent all reflexive rank 1 modules. The goal of the last section is to show that $\mathfrak{p}^k$ (resp. $\mathfrak{q}^k$) is a Cohen-Macaulay ideal if and only if $k \leq m-r$ (resp. $k \leq n-r$). In addition we prove that the powers $\mathfrak{p}^{m-r}$ and $\mathfrak{q}^{n-r}$ are even Ulrich ideals.

2. STANDARD BITABLEAUX

Let K be a field. For the study of the determinantal rings R_{r+1} we use the approach of standard bitableaux for which one considers all minors of the matrix X as generators for the K-algebra $K[X]$ and not only the 1-minors X_{ij}. Hence products of minors appear as "monomials".

Let $1 \le t \le \min\{m,n\}$. Denote the determinant of the matrix $X' = (X_{a_i b_j} : i = 1,\dots,t,$ $j = 1,\dots,t)$ by

$$[a_1 \dots a_t | b_1 \dots b_t].$$

We require that $1 \le a_1 < \cdots < a_t \le m$ and $1 \le b_1 < \cdots < b_t \le n$. We call $[a_1 \dots a_t | b_1 \dots b_t]$ a *minor* of X and t its *size*. A *bitableau* Δ is a product of minors

$$\prod_{i=1}^{w} [a_{i1} \dots a_{it_i} | b_{i1} \dots b_{it_i}] \quad \text{such that } t_1 \ge \cdots \ge t_w.$$

By convention the value of the empty minor $[\,|\,]$ is 1. The *shape* of Δ is the sequence $(t_1,\dots,t_w)$. The name bitableau is motivated by the graphical description of Δ as a pair of so-called Young tableaux, and we will also write $\Delta = (a_{ij} \,|\, b_{ij})$. We consider a partial order on the set of all bitableau:

$$[a_1 \dots a_t | b_1 \dots b_t] \preceq [c_1 \dots c_u | d_1 \dots d_u] \\ \iff \quad t \ge u \text{ and } a_i \le c_i,\ b_i \le d_i,\ i = 1,\dots,u.$$

A product $\Delta = \delta_1 \cdots \delta_w$ of minors $\delta_i = [a_{i1} \dots a_{it_i} | b_{i1} \dots b_{it_i}]$ is a *standard bitableau* if

$$\delta_1 \preceq \cdots \preceq \delta_w,$$

i. e. in each "column" of the bitableau the indices are non-decreasing from top to the bottom. (The empty product is also standard.) The letter Σ is reserved for standard bitableaux. The fundamental straightening law of Doubilet-Rota-Stein [DRS] says that every element of $K[X]$ has a unique presentation as a K-linear combination of standard bitableaux. Hence these elements are a K-vector space basis of $K[X]$ and $K[X]$ is an algebra with straightening law (ASL for short) on the set of standard bitableaux. See [BV] or Bruns and Conca [BC] for a detailed introduction.

We let $\mathscr{S}_r$ denote the set of all standard bitableaux whose left tableau has entries in $\{1,\dots,m\}$, whose right tableau has entries in $\{1,\dots,n\}$, and whose shape $(s_1,\dots,s_u)$ is bounded by the condition $s_1 \le r$.

For a (standard) bitableau Σ and an $m \times n$ matrix $A = (a_{ij})$ over some K-algebra B we let Σ_A denote the image of Σ under the homomorphism $K[X] \to B$ defined by the substitution $X_{ij} \mapsto a_{ij}$. However, for simplicity we will not explicitly indicate the passage from $K[X]$ to its residue class ring R_{r+1}.

Theorem 2.1. *The (residue classes of the) standard bitableaux $\Sigma \in \mathscr{S}_r$ generate R_{r+1} as a vector space over K.*

The proof of this theorem, essentially due to Hodge, is to be found in many sources. It is most easily proved by dehomogenization of its companion result for the subalgebra of $K[X]$ spanned by the maximal minors; for example, see [BV].

3. Initial algebras

The classical "generic point" for R_{r+1} is the homomorphism

$$\varphi : R_{r+1} \to K[Y,Z]$$

where Y is an $m \times r$ matrix of indeterminates, Z is an $r \times n$ matrix of indeterminates, and the homomorphism is induced by the substitution of the (i,j)-th entry of the product YZ for X_{ij}. The homomorphism $K[X] \to K[Y,Z]$ factors through R_{r+1} since $\operatorname{rank}(YZ) = r$.

On $K[Y,Z]$ we introduce a term order by first listing the variables of Y column by column from bottom to top, starting with the first column, and then the entries of Z row by row from right to left:

$$Y_{m1} > Y_{m-11} > \cdots > Y_{11} > Y_{m2} > \cdots > Y_{1r} > Z_{1n} > \cdots > Z_{11} > Z_{2n} > \cdots > Z_{r1}.$$

This total order is then extended to the induced degree reverse lexicographic order on $K[Y,Z]$. Note that the restrictions of the term orders to $K[Y]$ and $K[Z]$ are diagonal: the initial term of a minor of Y or Z is the product of its main diagonal elements. But also the initial monomials of the minors of YZ are easily found:

Lemma 3.1. *Let $1 \le t \le r$. The initial monomial of the minor $[a_1 \dots a_t \mid b_1 \dots b_t]_{YZ}$ is the monomial $Y_{a_1 1} \cdots Y_{a_t t} Z_{1 b_1} \cdots Z_{t b_t}$.*

Proof. Suppose first that $t = r$. Then the matrix $X' = (X_{a_i b_j})$ is the product of $Y' = (Y_{a_i j})$ and $Z' = (Z_{i b_j})$. Clearly

$$\operatorname{in}(\det(X')) = \operatorname{in}(\det(Y'Z')) = \operatorname{in}(\det(Y')\det(Z')) = \operatorname{in}(\det(Y'))\operatorname{in}(\det(Z')),$$

and the last term is the product of the main diagonals, as pointed out above.

Let now $t < r$. Since we have chosen the reverse lexicographic term order, we may delete all monomials from $[a_1 \dots a_t \mid b_1 \dots b_t]_{YZ}$ that involve an indeterminate Z_{ij} with $i > t$ without loosing the initial monomial, provided at least one term survives. But this is clearly the case: under the substitution $Z_{ij} \mapsto 0$ for $i > t$ the minor $[a_1 \dots a_t \mid b_1 \dots b_t]_{YZ}$ goes to the minor $[a_1 \dots a_t \mid b_1 \dots b_t]_{\overline{YZ}}$ where $\overline{Y}$ consists of the first t columns of Y and $\overline{Z}$ consists of the first t rows of Z. Now we have reached the case of maximal minors discussed above. $\square$

Proposition 3.2.

(a) *The initial monomial of the standard bitableau $\Sigma = (a_{ij} \mid b_{ij})$, $i = 1, \dots, u$, $j = 1, \dots, t_i$, $t_1 \ge \dots \ge t_u$ is the monomial $\prod_{i=1}^{u} \prod_{j=1}^{t_i} Y_{a_{ij} j} Z_{j b_{ij}}$.*

(b) *If $\Sigma, \Sigma' \in \mathscr{S}_r$, $\Sigma \ne \Sigma'$, then $\operatorname{in}(\Sigma_{YZ}) \ne \operatorname{in}(\Sigma'_{YZ})$. In particular, the polynomials Σ_{YZ} are K-linearly independent.*

Proof. Part (a) is an immediate consequence of Lemma 3.1. For part (b) one observes that the factors Y_{vw} that appear in $\operatorname{in}(\Sigma_{YZ})$ uniquely determine the w-th column of the left tableau of Σ since they indicate which indices v appear in this column and determine their multiplicities. The indices in a column are non-decreasing (from top to bottom), and therefore the column is uniquely given by the indices and their multiplicities. It follows that the left tableau is uniquely determined, and a similar argument applies to the right tableau. The linear independence follows immediately. $\square$

We draw a well-known consequence.

Corollary 3.3. *Let $K[YZ]$ denote the K-algebra generated by the entries of the product matrix YZ.*

(a) *The homomorphism $\varphi : R_{r+1} \to K[YZ]$ is an isomorphism.*
(b) *The standard bitableaux $\Sigma \in \mathscr{S}_r$ form a K-basis of R_{r+1}.*

In fact, the homomorphism maps the elements of a system of generators of the vector space R_{r+1} to a linearly independent system in its image $K[YZ]$. In the following we will identify R_{r+1} with $K[YZ]$.

Remark 3.4. (a) The above proof of the *straightening law* contained in 3.3(b) can be used for an effective implementation as follows. Given an element $f \in R_{r+1}$ (so $f \in K[X]$ if $r = \min(m,n)$), we map it to $K[YZ]$. Then the initial term of $\varphi(f)$ is determined. It determines a unique standard monomial Σ. Next Σ is evaluated in R_{r+1} (of course, not in $K[YZ]$!), and we replace f by $f - \lambda\Sigma$ where λ is the leading coefficient of $\varphi(f)$. Since $f - \lambda\Sigma = 0$ or $\operatorname{in}(\varphi(f-\lambda\Sigma)) < \operatorname{in}(\varphi(f))$, an iteration of the procedure must terminate after finitely many steps.

(b) In order to avoid Theorem 2.1 in the proof of the straightening law one would have to show that the initial monomial of an arbitrary element in $K[YZ]$ is one of the monomials $\operatorname{in}(\Sigma_{YZ})$, $\Sigma \in \mathscr{S}_r$.

(c) If one is willing to invest the Knuth-Robinson-Schensted correspondence, then Theorem 2.1 becomes a consequence of Proposition 3.2: the correspondence implies that in each degree there exist as many standard bitableaux in $\mathscr{S}_t$, $t = \min(m,n)$, as ordinary monomials. Together with the linear independence of $\mathscr{S}_t$ (in whose proof Theorem 2.1 has not been used), this implies that $\mathscr{S}_t$ is a K-basis of $K[X]$. This shows 2.1 for $r = t$. The general case follows rapidly since we have the inclusions

$$V_{r+1} \subset I_{r+1}(X) \subset \operatorname{Ker}(\varphi)$$

where V_{r+1} is the vector space spanned by all $\Sigma \notin \mathscr{S}_r$ and $\mathscr{S}_r$ is mapped to a linear independent subset of $K[YZ]$. (Note that every minor of size $> r$ is contained in I_{r+1}.)

(d) We will show that the initial algebra of R_{r+1} is a normal semigroup ring. This is a direct generalization of the fact that for $r = 1$ the algebra $R_2 = K[YZ] = D_2$ is a normal semigroup ring itself.

We are in the extremely rare situation that taking initial forms on a vector space basis is injective, and so we can immediately describe the initial space:

Theorem 3.5.

(a) *The initial algebra $D_{r+1} = \operatorname{in}(R_{r+1}) \subset K[Y,Z]$ is generated by the monomials $Y_{a_1 1} \cdots Y_{a_t t} Z_{1 b_1} \cdots Z_{t b_t}$ with $1 \le t \le r$, $a_1 < \cdots < a_t$ and $b_1 < \cdots < b_t$.*
(b) *D_{r+1} is a normal semigroup ring.*
(c) *R_{r+1} is a normal domain, Cohen-Macaulay, with rational singularities in characteristic 0, and F-rational in characteristic $p > 0$.*

Proof. (a) This is just a reformulation of Proposition 3.2. In fact, the subalgebra generated by the monomials given in (a) is a K-vector subspace of D_{r+1}. On the other hand, it has the same Hilbert function as R_{r+1} (or D_{r+1}). This forces equality.

(b) It is enough to show that $M^k \in D_{r+1}$ for a monomial $M \in K[Y,Z]$ and an integer $k > 0$ implies $M \in D_{r+1}$. There exists a standard bitableau $\Sigma = (a_{ij} \,|\, b_{ij})$ with $M^k = \operatorname{in}(\Sigma)$. We then write M^k in the form $\prod_{i=1}^{u}\prod_{j=1}^{t_i} Y_{a_{ij}j}Z_{jb_{ij}}$. Since M^k is a k-th power and Σ is a standard bitableau, the first factor $\prod_{j=1}^{t_1} Y_{a_{1j}j}Z_{jb_{1j}}$ must occur (at least) k times. We split it off M, and conclude by induction.

(c) follows from general theorems on flat deformation. For proofs see [BC] or Conca, Herzog and Valla [CHV]. □

The Cohen-Macaulay property of R_{r+1} was first proved by Hochster and Eagon [HE] and the Cohen-Macaulay property of normal semigroup rings by Hochster [Ho].

Remark 3.6. For an application below we describe the set E of vectors $[(\alpha_{ij}),(\beta_{uv})] \in (\mathbb{R}^{mr}) \oplus (\mathbb{R}^{rn})$ that appear as exponent vectors of elements in $D_{r+1} = \operatorname{in}(R_{r+1})$. It is not hard to check that E is the set of lattice points in the cone defined by the following linear equations and inequalities:

$$\alpha_{ij} = \beta_{uv} = 0, \qquad j > i,\ u > v, \tag{1}$$

$$\sum_{i=j-1}^{k-1} \alpha_{ij-1} - \sum_{i=j}^{k} \alpha_{ij} \geq 0, \qquad j = 2,\dots,r,\ \ k = j,\dots,m, \tag{2}$$

$$\sum_{t=u-1}^{w-1} \beta_{u-1t} - \sum_{t=u}^{w} \beta_{ut} \geq 0, \qquad t = 2,\dots,r,\ \ w = u,\dots,n, \tag{3}$$

$$\alpha_{ij}, \beta_{uv} \geq 0, \qquad i > j,\ v > u \ \text{ and } \ i = j = u = v = r, \tag{4}$$

$$\sum_{i=1}^{n} \alpha_{ij} - \sum_{v=1}^{n} \beta_{jv} = 0 \qquad j = 1,\dots,r. \tag{5}$$

Note that for $r = \min(m,n)$ we consider an embedding of $K[X]$ into $K[Y,Z]$ which identifies the indeterminate X_{ij} with the corresponding entry of the product matrix YZ. Thus we can investigate the initial ideal $\operatorname{in}(J) \subset D = \operatorname{in}(K[X])$ for every ideal J of $K[X]$. In particular, it is useful to consider the ideals $I(X;\delta)$ and the residue class rings $R(X;\delta) = K[X]/I(X;\delta)$ where $I(X;\delta)$ is generated by all minors $\gamma \not\geq \delta$. Observe that $R_{r+1} = R(X;\delta)$ for $\delta = [1\dots r \,|\, 1\dots r]$. The proof of the next corollary shows that we recover D_{r+1} as a retract of D if we take $\delta = [1\dots r \,|\, 1\dots r]$.

Corollary 3.7. *Let D be the initial algebra of $K[X]$. The initial ideal $\operatorname{in}(I(X;\delta))$ is a (monomial) prime ideal in D. Therefore $R(X;\delta)$ is a normal Cohen-Macaulay domain with rational singularities in characteristic 0, and F-rational in characteristic p.*

Proof. Let $\delta = [a_1 \dots a_t \,|\, b_1 \dots b_t]$ and $\gamma = [c_1 \dots c_u \,|\, d_1 \dots d_u]$. Then $\gamma \not\geq \delta$ if $u > t$ or $c_i < a_i$ or $d_i < b_i$ for some $i = 1,\dots,u$. Thus $\operatorname{in}(I(X;\delta))$ is generated by those monomials for which certain exponents are positive. This shows that $J = \operatorname{in}(I(X;\delta))$ is a prime ideal.

Therefore the residue class ring D/J is (isomorphic to) a normal semigroup ring: D/J is a retract of D. Now the deformation arguments apply again. □

Let $\mathrm{GL} = \mathrm{GL}(r,K)$ be the general linear group of invertible $r \times r$-matrices with entries in K. For $f(Y,Z) \in K[Y,Z]$ and $T \in \mathrm{GL}$ we set $T(f) = f(YT^{-1}, TZ)$. This defines a group

action on $K[Y,Z]$ as a group of K-automorphisms on $K[Y,Z]$. It turns out that if $|K| = \infty$, then $K[YZ] \cong R_{r+1}$ is the ring of invariants $K[Y,Z]^{\mathrm{GL}}$ under the action of GL. In the general case one can show that $K[YZ]$ is the ring of the so-called absolute GL-invariants.

Similar one can consider the action of the special linear group $\mathrm{SL} = \mathrm{SL}(r,K) = \{T \in \mathrm{GL}(r,K)\colon \det(T) = 1\}$ on $K[X,Y]$. In this case the ring of (absolute) SL-invariants is the K-subalgebra $\tilde{R}_{r+1} \subset K[Y,Z]$ generated by the entries of YZ, the r-minors of Y and the r-minors of Z. (See [BV], Section 7 for definitions and proofs.) We can study the ring $\tilde{R}_{r+1}$ analogously to R_{r+1}.

Theorem 3.8.

(a) *The initial algebra* $\tilde{D}_{r+1} = \mathrm{in}(\tilde{R}_{r+1}) \subset K[Y,Z]$ *is generated by the monomials*
  (i) $Y_{a_1 1} \cdots Y_{a_t t} Z_{1 b_1} \cdots Z_{t b_t}$ *with* $1 \le t < r$, $a_1 < \cdots < a_t$ *and* $b_1 < \cdots < b_t$,
  (ii) $Y_{a_1 1} \cdots Y_{a_r r}$ *with* $a_1 < \cdots < a_r$,
  (iii) $Z_{1 b_1} \cdots Z_{r b_r}$ *with* $b_1 < \cdots < b_r$.

(b) $\tilde{D}_{r+1}$ *is a normal semigroup ring.*

(c) $\tilde{R}_{r+1}$ *is a normal domain, Cohen-Macaulay, with rational singularities in characteristic* 0, *and* F*-rational in characteristic* $p > 0$.

Proof. Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the ideal of $K[YZ] \cong R_{r+1}$ generated by the set Γ_r (resp. Γ_c) consisting of all r-minors of the first r rows (resp. the first r columns) of the matrix YZ. We investigate the ideals $\mathfrak{p}^t$ and $\mathfrak{q}^t$. The set of all standard bitableaux, which contain at least t factors of Γ_r (resp. Γ_c) form a K-basis of $\mathfrak{p}^t$ (resp. $\mathfrak{q}^t$). (This follows directly from the fact that $\mathfrak{p}$ and $\mathfrak{q}$ are straightening-closed ideals of $K[YZ]$; compare [BV, 9.6].)

$K[Y,Z]$ is a bigraded K-algebra in which all entries of Y have bidegree $(1,0)$ and all entries of Z have bidegree $(0,1)$. Note that $\tilde{R}_{r+1}$ is a graded K-subalgebra of $K[Y,Z]$ where $(\tilde{R}_{r+1})_t$ contains the bihomogeneous elements (d_1,d_2) such that $d_2 - d_1 = tr$. In [BV, 9.21] it is shown that $(\tilde{R}_{r+1})_t$ is isomorphic to $\mathfrak{p}^t$ as a K-vector space if $t \ge 0$ and isomorphic to $\mathfrak{q}^{-t}$ as a K-vector space if $t \le 0$. This isomorphism is induced by

$$[a_1 \dots a_r]_Y \mapsto [a_1 \dots a_r \,|\, 1 \dots r]_{YZ}, \quad [b_1 \dots b_r]_Z \mapsto [1 \dots r \,|\, b_1 \dots b_r]_{YZ};$$

Observe that $[a_1 \dots a_r]_Y [b_1 \dots b_r]_Z = [a_1 \dots a_r \,|\, b_1 \dots b_r]_{YZ}$. Then a K-basis of $\tilde{R}_{r+1}$ consists of the monomials

$$\prod_{i=1}^{t_1} [a_{i1} \dots a_{ir}]_Y \cdot \Sigma_1, \quad \prod_{i=1}^{t_2} [b_{i1} \dots b_{ir}]_Z \cdot \Sigma_2$$

where Σ_1, Σ_2 are standard monomials in $K[YZ] \cong R_{r+1}$ and

$$\prod_{i=1}^{t_1} [a_{i1} \dots a_{ir} \,|\, 1 \dots r]_{YZ} \cdot \Sigma_1, \quad \prod_{i=1}^{t_2} [1 \dots r \,|\, b_{i1} \dots b_{ir}]_{YZ} \cdot \Sigma_2$$

are standard monomials in $\mathfrak{p}^{t_2}$ (resp. $\mathfrak{q}^{t_1}$). It follows from 3.2 and the observation before that the initial monomials are

$$\operatorname{in}\left(\prod_{i=1}^{t_1}[a_{i1},\dots,a_{ir}]_Y\cdot\Sigma_1\right)=\prod_{i=1}^{t_1}Y_{a_{i1}1}\cdots Y_{a_{ir}r}\cdot\operatorname{in}(\Sigma_1),$$
$$\operatorname{in}\left(\prod_{i=1}^{t_2}[b_{i1},\dots,b_{ir}]_Z\cdot\Sigma_2\right)=\prod_{i=1}^{t_2}Z_{1b_{i1}}\cdots Z_{rb_{ir}}\cdot\operatorname{in}(\Sigma_2).$$

These distinct monomials are a K-basis of $\tilde{D}_{r+1}$, since the Hilbert functions of $\tilde{D}_{r+1}$ and $\tilde{R}_{r+1}$ coincide. This already proves (a).

To prove (b) one argues similar to the proof of 3.5, and (c) follows again from general theorems on flat deformation. □

Remark 3.9. Again we can describe the set $\tilde{E}$ of vectors $[(\alpha_{ij}),(\beta_{uv})]\in(\mathbb{R}^{mr})\oplus(\mathbb{R}^{rn})$ that appear as exponent vectors of elements in $\tilde{D}_{r+1}=\operatorname{in}(\tilde{R}_{r+1})$. It is the set of lattice points in the cone defined by the conditions (1)–(4) and

$$\sum_{i=1}^{n}\alpha_{ij}-\sum_{v=1}^{n}\beta_{jv}=0,\qquad j=1,\dots,r-1.$$

Note that we have left out exactly one equation from (5), namely that for $j=r$.

Remark 3.10. (a) The program by which Theorems 3.5 and 3.8 have been proved consists of three steps: (1) determine the initial algebra $\operatorname{in}(R)$ of an algebra R (with respect to a suitable embedding of R into a polynomial ring), (2) show that $\operatorname{in}(R)$ is normal and (3) conclude that R is normal, Cohen-Macaulay, with rational singularities in characteristic 0, and F-rational in characteristic $p>0$.

This program can also be carried out for several objects derived from or similar to the rings R_{r+1}:

(i) The Rees algebra $\bigoplus_k\bar{I}_{s+1}^kT^k\subset R_{r+1}[T]$ where $s<r$ and $\bar{I}_{s+1}$ is the ideal generated by the residue classes of the $s+1$-minors in R_{r+1}.
(ii) The symbolic Rees algebra $\bigoplus_k\bar{I}_{s+1}^{(k)}T^k\subset R_{r+1}[T]$ where $\bar{I}_{s+1}^{(k)}$ is the symbolic powers of $\bar{I}_{s+1}$.
(iii) The subalgebra $A_{r+1,t}$ of R_{r+1} which is generated by the residue classes of all t-minors of the matrix X.

For (i) and (iii) one needs that the characteristic of K is 0 or $>\min\big(s+1,m-(s+1),n-(s+1)\big)$; see [BV, Section 10] or [BC].

(b) One can also consider a symmetric $n\times n$-matrix X^{sym} of indeterminates, i.e. $X_{ij}^{\mathrm{sym}}=X_{ji}^{\mathrm{sym}}$. In this situation we have to replace the generic point $K[YZ]$ of $K[X]$ above with the generic point $K[YY^{\mathrm{tr}}]$ of $K[X^{\mathrm{sym}}]$ where Y is an $n\times r$ matrix and Y^{tr} is the transpose of Y. The proofs are almost the same with minor modifications.

(c) The method presented above provides a comfortable approach to the structural properties of the determinantal rings. Despite of the fact that we use term orders it is not a substitute for the computation of *Gröbner bases* of the determinantal ideals *within* $K[X]$, or, more precisely, with respect to the monoid of monomials of $K[X]$. For this task one

has to use other methods, for example the Knuth-Robinson-Schensted correspondence. (See [BC] for details.)

4. Cohen-Macaulay and Ulrich ideals

Suppose that $1 \le r < \min\{m,n\}$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the ideals in R_{r+1} as defined in the proof of Theorem 3.8: $\mathfrak{p}$ is generated by the r-minors of the first r rows and $\mathfrak{q}$ is generated by the r-minors of the first r columns.

Let J be a reflexive rank 1 module. Then J is isomorphic to a divisorial ideal. It is known that the classes $[\mathfrak{p}],[\mathfrak{q}] \in \mathrm{Cl}(R_{r+1})$ are inverse to each other and that each of them generates (the infinite cyclic group) $\mathrm{Cl}(R_{r+1})$; e.g. see [BV, (8.4)]. This implies that all divisorial ideals are represented by the symbolic powers $\mathfrak{p}^{(t)}$ and $\mathfrak{q}^{(t)}$, $t \ge 0$. Moreover, $\mathfrak{p}^{(t)} = \mathfrak{p}^t$ and $\mathfrak{q}^{(t)} = \mathfrak{q}^t$ for all t [BV, (9.18)]. Thus $J \cong \mathfrak{p}^t$ or $J \cong \mathfrak{q}^t$ for some $t \ge 0$. Hence, up to isomorphism the powers $\mathfrak{p}^t$ and $\mathfrak{q}^t$ represent all reflexive rank 1 modules. In this section we study their Cohen-Macaulay and Ulrich property.

We briefly recall the definition of an Ulrich ideal: Let S be a homogeneous Cohen-Macaulay K-algebra and let M be a finitely generated graded maximal Cohen-Macaulay S-module. Then $\mu(M) \le e(M)$ where $\mu(M)$ denotes the minimal number of generators of M and $e(M)$ denotes the multiplicity of M (e.g. see Brennan, Herzog and Ulrich [BHU]). In case of equality, M is called an *Ulrich module*. A graded ideal $I \subset S$ is said to be an *Ulrich ideal* if it is an Ulrich module. If S is a domain and $I \ne 0$ then $e(I) = e(S)$ and hence I is an Ulrich ideal if and only if it is Cohen-Macaulay and $\mu(I) = e(S)$.

We start by computing the minimal number of generators for the powers of the ideals $\mathfrak{p}$ and $\mathfrak{q}$.

Proposition 4.1. *For any integer $t \ge 1$ the number $\mu(\mathfrak{p}^t)$ is equal to the determinant of the matrix*

$$\left[\binom{t+n-j}{n-i}\right]_{1\le i,j\le r}$$

and the number $\mu(\mathfrak{q}^t)$ is equal to the determinant of the matrix

$$\left[\binom{t+m-j}{m-i}\right]_{1\le i,j\le r}.$$

Proof. By symmetry it is enough to prove the assertion for $\mathfrak{p}^t$. According to [BV, (9.3)] the ideal $\mathfrak{p}^t$ is generated by the standard bitableaux which are products of exactly t r-minors of the first r rows of X (modulo I_{r+1}). These standard bitableaux are K-linearly independent. Their number coincides with the number of standard bitableaux with t factors in the coordinate ring $G(r,n)$ of the Grassmannian of r-dimensional vector spaces in K^n because the latter elements are the preimages of the generators of $\mathfrak{p}^t$ in $K[X]$. So we can finish our proof quoting the classical formula of Hodge (for example, see Ghorpade [Gh, Theorem 6]) by which $\dim_K G(r,n)_t$ is equal to the determinant of the matrix given in the assertion. $\square$

Next we show:

Lemma 4.2. *The multiplicity of R_{r+1} coincides with $\mu(\mathfrak{p}^{m-r})$ and $\mu(\mathfrak{q}^{n-r})$.*

Proof. The multiplicity of R_{r+1} is known to be the determinant of the matrix

$$B = \left[\binom{m+n-i-j}{n-j}\right]_{1\le i,j\le r}.$$

(E.g. see Herzog and Trung [HT].) By the above proposition we know that $\mu(\mathfrak{p}^{m-r})$ is equal to the determinant of the matrix

$$A = \left[\binom{m+n-r-j}{n-i}\right]_{1\le i,j\le r}.$$

Using the binomial identity $\binom{a}{b}+\binom{a}{b+1}=\binom{a+1}{b+1}$ one can transform A into the transpose of B by elementary row operations which do not affect the determinant. This proves $\mu(\mathfrak{p}^{m-r}) = e(R_{r+1})$. The equation $\mu(\mathfrak{q}^{n-r}) = e(R_{r+1})$ can be obtained in an analogous way. $\square$

As a function of t the minimal number of generators $\mu(\mathfrak{p}^t)$ is evidently a strictly increasing function in t. Thus $\mu(\mathfrak{p}^t) > \mu(\mathfrak{p}^{m-r}) = e(R_{r+1})$ for $t > m-r$, and $\mathfrak{p}^t$ cannot be a Cohen-Macaulay ideal. By the same reason $\mathfrak{q}^t$ cannot be Cohen-Macaulay for $t > n-r$.

Theorem 4.3. *Let $t \ge 1$ be an integer. The power $\mathfrak{p}^t$ (resp. $\mathfrak{q}^t$) is a Cohen-Macaulay ideal if and only if $t \le m-r$ (resp. $t \le n-r$). The powers $\mathfrak{p}^{m-r}$ and $\mathfrak{q}^{n-r}$ are both Ulrich ideals.*

Proof. The crucial point which has not been proved yet is that $\mathfrak{p}^t$ (resp. $\mathfrak{q}^t$) is a Cohen-Macaulay ideal for $t \le m-r$ (resp. $t \le n-r$). By symmetry it is enough to deal with $\mathfrak{p}^t$.

Assume that $t \le m-r$. We consider the set of all standard bitableaux of R_{r+1}, which contain at least t factors of the generators of $\mathfrak{p}$. We already observed in the proof of 3.8 that these elements form a K-basis of $\mathfrak{p}^t$. Now we use the generic point $\varphi : R_{r+1} \to K[Y,Z]$ to embed $\mathfrak{p}^t$ into $K[Y,Z]$, and investigate the initial ideal $\mathfrak{a}_t = \operatorname{in}(\varphi(\mathfrak{p}^t)) \subset D_{r+1}$.

Let E_t be the subset of E (compare 3.6 for the definition of E) consisting of all vectors in $(\mathbb{R}^{mr}) \oplus (\mathbb{R}^{rn})$ that appear as exponent vectors of the elements in $\mathfrak{a}_t$. One easily checks that

$$\begin{aligned} E_t &= \{\, [(\alpha_{ij}),(\beta_{uv})] \in E \mid \alpha_{ii} \ge t,\ \ i = 1,\dots,r \,\} \\ &= \{\, [(\alpha_{ij}),(\beta_{uv})] \in E \mid \alpha_{rr} \ge t \,\}. \end{aligned}$$

We want to show that $\mathfrak{a}_t$ is a conic ideal in D_{r+1} (see Bruns and Gubeladze [BG, Section 3]). To this end we have to find $w_t \in \mathbb{R}E$ such that $E_t = \mathbb{Z}E \cap (w_t + \mathbb{R}_+E)$. Note that $\mathbb{R}E$ is the set of all vectors $[(\alpha_{ij}),(\beta_{uv})] \in (\mathbb{R}^{mr}) \oplus (\mathbb{R}^{rn})$ that satisfy the equations

$$\alpha_{ij} = \beta_{uv} = 0, \quad j > i, u > v, \qquad \sum_{i=1}^{n} \alpha_{ij} - \sum_{v=1}^{n} \beta_{jv} = 0, \quad j = 1,\dots,r,$$

and that $\mathbb{Z}E = \mathbb{R}E \cap ((\mathbb{Z}^{mr}) \oplus (\mathbb{Z}^{rn}))$. We choose a positive real number $\varepsilon < 1$ and define $w_t = [(\alpha_{ij}),(\beta_{uv})]$ by setting

$$\alpha_{ij} = \begin{cases} t-\varepsilon, & \text{if } i = j, \\ -(t-\varepsilon)/(m-r), & \text{if } j < i \le m-r+j, \\ 0, & \text{otherwise.} \end{cases}$$

and $\beta_{uv} = 0$ for all u, v. It is clear that $w_t \in \mathbb{R}E$. Since $-(t-\varepsilon)/(m-r) > -1$ (this is the point where we need $t \le m-r\,!$) we have $\mathbb{Z}E \cap (w_t + \mathbb{R}_+ E) = E_t$. So $\mathfrak{a}_t$ is indeed a conic ideal. Since every conic ideal in a normal semigroup ring is Cohen-Macaulay (see [BG, 3.3]) we conclude that $\mathfrak{a}_t$ is a Cohen-Macaulay ideal in the ring D_{r+1}. But this implies that $\mathfrak{p}^t$ is a Cohen-Macaulay ideal in the ring R_{r+1} (e.g. see [BC, 3.16]). □

The case $r = 1$ of the theorem has been proved (and the general has been conjectured) by Bruns and Guerrieri [BrGu].

Corollary 4.4. *The ideals $\mathfrak{p}^t$, $0 \le t \le m-r$, and $\mathfrak{q}^t$, $0 < t \le n-r$, represent all isomorphism classes of maximal Cohen-Macaulay R_{r+1}-modules of rank* 1.

Proof. Let M be a maximal Cohen-Macaulay R_{r+1}-module of rank 1. Then M is torsion-free and therefore it is isomorphic to a fractionary ideal J of R_{r+1}. Using the reflexivity criterion of [BH, 1.4.1], one sees that J is reflexive and hence divisorial.

We already noticed in the beginning of this section that then $J \cong \mathfrak{p}^t$ or $J \cong \mathfrak{q}^t$ for some $t \ge 0$, and the corollary follows immediately from Theorem 4.3. □

REFERENCES

[BHU] J. P. Brennan, J. Herzog and B. Ulrich, *Maximally generated Cohen-Macaulay modules*. Math. Scand. **61** (1987), 181–203.

[BC] W. Bruns and A. Conca, *Gröbner bases and determinantal ideals*. In: J. Herzog and V. Vuletescu (eds.), Commutative Algebra, Singularities and Computer Algebra. Kluwer, 2003, pp. 9–66.

[BG] W. Bruns and J. Gubeladze, *Divisorial linear algebra of normal semigroup rings*. Algebr. Represent. Theory **6** (2003), 139–168.

[BrGu] W. Bruns and A. Guerrieri, *The Dedekind-Mertens formula and determinantal ideals*. Proc. Amer. Math. Soc. **127** (1999), 657–663.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*. Rev. ed. Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1998.

[BV] W. Bruns and U. Vetter, *Determinantal rings*. Lect. Notes Math. **1327**, Springer, 1988.

[CHV] A. Conca, J. Herzog and G. Valla, *Sagbi bases and application to blow-up algebras*. J. Reine Angew. Math. **474** (1996), 113–138.

[DRS] P. Doubilet, G.-C- Rota and J. Stein, *On the foundations of combinatorial theory: IX, Combinatorial methods in invariant theory*. Stud. Appl. Math. **53** (1974), 185–216.

[Gh] S. R. Ghorpade, *A Note on Hodge's Postulation Formula for Schubert varieties*, In J. Herzog and G. Restuccia (eds.), *Geometric and combinatorial aspects of commutative algebra*, Lect. Notes in Pure and Appl. Math. **217**, M. Dekker, 211–219, 2001.

[GL] N. Gonciulea, N. and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*. Transform. Groups **1** (1996), 215–248.

[HT] J. Herzog and N. V. Trung, *Gröbner Bases and Multiplicity of Determinantal and Pfaffian ideals*, Adv. Math. **96** (1992), 1–37.

[Ho] M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes.* Ann. Math. **96** (1972), 318–337.

[HE] M. Hochster and J.A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci.* Amer. J. Math. **93** (1971), 1020–1058.

[St] B. Sturmfels, *Algorithms in invariant theory*. Springer (Wien) 1993.

FB MATHEMATIK/INFORMATIK, UNIVERSITÄT OSNABRÜCK, 49069 OSNABRÜCK, GERMANY
E-mail address: winfried@mathematik.uni-osnabrueck.de

FB MATHEMATIK/INFORMATIK, UNIVERSITÄT OSNABRÜCK, 49069 OSNABRÜCK, GERMANY
E-mail address: troemer@mathematik.uni-osnabrueck.de

FB MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, 45117 ESSEN, GERMANY
E-mail address: attila.wiebe@ruhrgas.com